A physicalist account of mathematical truth

tea talk at Wigner
David Nagy
• What are mathematical entities?
• What does it mean to refer to a mathematical object?
• What is the relation between logic and mathematics?
• What kinds of inquiry play a role in mathematics?
• What are the objectives of mathematical inquiry?
• What gives mathematics its hold on experience?
• What is the source and nature of mathematical truth?
• What is the relationship between the abstract world of mathematics and the material universe?
platonist interpretation of mathematics
Platonism is the view that mathematics describes a non-sensual reality, which exists independently both of the acts and [of] the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind.

Gödel 1995, p. 323
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Gödel 1995, p. 323

Realism or platonism is the view that mathematics is the scientific study of objectively existing mathematical entities just as physics is the study of physical entities. The statements of mathematics are true or false depending on the properties of those entities, independent of our ability, or lack thereof, to determine which.

Maddy 1990, p. 21
Penrose
formalist interpretation of mathematics
“The formulas are not about anything; they are just strings of symbols”
“The formulas are not about anything; they are just strings of symbols”

“One must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs”

Hilbert (allegedly)
mathematics is a formal system
what is a formal system?
formal system
formal system

symbols
formal system

symbols

axioms
formal system

- symbols
- axioms
- inference rules
chess is a formal system
symbols

{♖, ♔, ♕, ♗, ♘, ♙, ♜, ♚, ♛, ♝, ♞, ♟}
symbols
\{ 
\text{♖}, \text{♘}, \text{♗}, \text{♕}, \text{♔}, \text{♙}, \text{♜}, \text{♚}, \text{♛}, \text{♝}, \text{♞}, \text{♟} \}

axioms
\text{♖} \text{♘} \text{♗} \ldots \text{♙} \text{♙} \text{♙} \text{♙} \text{♙} \text{♙} \text{♙} \text{♙} \text{♙} \text{♙} \text{♙} \text{♖} \text{♘} \text{♗}
symbols
\{\text{\text{Rook}}, \text{Queen}, \text{King}, \text{Bishop}, \text{Knight}, \text{Pawn}, \text{Rook}, \text{Queen}, \text{King}, \text{Bishop}, \text{Knight}, \text{Pawn}\}\}

axioms
\begin{tabular}{cccccccc}
\text{Rook} & \text{Knight} & \text{[...]} & \text{[...]} & \text{[...]} & \text{[...]} & \text{Bishop} & \text{Knight} \\
\end{tabular}

inference rules
\begin{tabular}{cccccccc}
\text{...} & \text{[...]} & \text{[...]} & \text{...} & \text{[...]} & \text{[...]} & \text{[...]} & \text{[...]} \\
\end{tabular}
chess is a formal system

formal system
- symbols
- axioms
- inference rules

chess
- pieces and fields
- starting positions
- rules of chess
a dynamical system is a formal system
a dynamical system is a formal system

formal system
symbols
axioms
inference rules
dynamical system
states
initial conditions
time evolution
game of life is a formal system
mathematics is a formal system
mathematics is a formal system

Alphabet

variables $x, y, z, \ldots$
individual constants $e$ (identity)
function symbols $i, p$ (inverse, product)
predicate symbol $=$
others $(, )$
logical symbols $\forall, \neg, \rightarrow$

Derivation rules

(MP) $\phi, (\phi \rightarrow \psi) \Rightarrow \psi$ (modus ponens)
(G) $\phi \Rightarrow \forall x \phi$ (generalization)

Axioms I. (logical)

(PC1) $(\phi \rightarrow (\psi \rightarrow \phi))$
(PC2) $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
(PC3) $((\neg \phi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \phi))$
(PC4) $(\forall x (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall x \psi))$ (given that $x$ is not free in $\phi$)
(PC5) $(\forall x \phi \rightarrow \phi)$ (given that $x$ is not free in $\phi$)
(PC6) $(\forall x \phi(x) \rightarrow \phi(y))$ (given that whenever a free occurrence of $x$ is replaced by $y$, $y$ is free in $\phi(y)$)

(E1) $x = x$
(E2) $t = s \rightarrow f^n (u_1, u_2, \ldots, t, \ldots, u_n) = f^n (u_1, u_2, \ldots, s, \ldots, u_n)$
(E3) $t = s \rightarrow (\phi (u_1, u_2, \ldots, t, \ldots, u_n) \rightarrow \phi (u_1, u_2, \ldots, s, \ldots, u_n))$

Axioms II. (of group theory)

(G1) $p(p(x, y), z) = p(x, p(y, z))$ (associative law)
(G2) $p(e, x) = x$ (left identity)
(G3) $p(i(x), x) = e$ (left inverse)
mathematics is a formal system

**Alphabet**

- variables: \( x, y, z, \ldots \)
- individual constants: \( e \) (identity)
- function symbols: \( i, p \) (inverse, product)
- predicate symbol: \( = \)
- others: \( (,), \to \)
- logical symbols: \( \forall, \neg \to \)

**Derivation rules**

- (MP) \( \phi, (\phi \rightarrow \psi) \Rightarrow \psi \) (modus ponens)
- (G) \( \phi \Rightarrow \forall x \phi \) (generalization)

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- (E1) \( x = x \)
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**Axioms II. (of group theory)**

- (G1) \( p(p(x, y), z) = p(x, p(y, z)) \) (associative law)
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truth in a physical theory
semantic truth vs mathematical truth
physical theory  ←  physical world
physical theory  \[\rightarrow\] physical world

formal system

\[ L \]

\[ a = \frac{dv}{dr} \]
\[ v = \frac{dr}{dt} \]
\[ r \]

Earth and Moon image
physical theory \[\xrightarrow{L} \text{formal system} \xrightarrow{S} \text{semantics} \xleftarrow{\text{physical world}}\]
A is a theorem if

\[ L \vdash A \]
A is **true** if according to semantic S, A refers to a state that is in fact the case.
A is **true** if according to semantic $S$ it refers to a state that is in fact the case

$L \vdash A \leftrightarrow S(A)$ is the state of the world
why be a formalist?
how do you know you are right?
how do you know you are right?

“gut feeling”
how do you know you are right?

$1 \cdot 1 \cdot 1 = 1$

“gut feeling”
how do you know you are right?

\[ 1 \cdot 1 \cdot 1 = 1 \]

“gut feeling”

I feel that

\[ p(e, p(e, e)) = e \]
how do you know you are right?

$1 \cdot 1 \cdot 1 = 1$

“gut feeling”

I feel that $p(e, p(e, e)) = e$

I also feel that $p(e, p(e, e)) = e$
how do you know you are right?

\[ 1 \cdot 1 \cdot 1 = 1 \]

“gut feeling”

I feel that
\[ p(e, p(e, e)) = e \]

I also feel that
\[ p(e, p(e, e)) = e \]

I strongly feel that
\[ p(e, p(e, e)) \neq e \]
• “Why is $p(e, p(e, e)) = e$ true?”
• “Why is $p(e, p(e, e)) = e$ true?”

• “How do we know that $p(e, p(e, e)) = e$ is true?”
• “Why is $p(e, p(e, e)) = e$ true?”

• “How do we know that $p(e, p(e, e)) = e$ is true?”

• “How can we verify that $p(e, p(e, e)) = e$ is true?”
• “Why is \( p(e, p(e, e)) = e \) true?”

• “How do we know that \( p(e, p(e, e)) = e \) is true?”

• “How can we verify that \( p(e, p(e, e)) = e \) is true?”

“In answering these questions, the mathematician never even mentions how the things are in the platonic world and never even mentions the epistemic means by which we have access to these realms. For the mathematician’s final argument is that formula \( p(e, p(e, e)) = e \) has a proof in group theory”
"In answering these questions, the mathematician never even mentions how the things are in the platonic world and never even mentions the epistemic means by which we have access to these realms. For the mathematician’s final argument is that formula $p(e, p(e, e)) = e$ has a proof in group theory”

(1) $p(e, x) = x$
(2) $(\forall x)(p(e, x) = x)$
(3) $(\forall x)(p(e, x) = x) \rightarrow p(e, e) = e$
(4) $p(e, e) = e$
(5) $(\forall x)(p(e, x) = x) \rightarrow p(e, p(e, e)) = p(e, e)$
(6) $p(e, p(e, e)) = p(e, e)$
(7) $p(e, e) = e \rightarrow p(e, p(e, e)) = p(e, e) \rightarrow p(e, p(e, e)) = e$
(8) $p(e, p(e, e)) = p(e, e) \rightarrow p(e, p(e, e)) = e$
(9) $p(e, p(e, e)) = e$

(G2) (G)
(PC6) (2), (3), (MP)
(2), (5), (MP)
(E3)
(4), (7), (MP)
(6), (8), (MP)
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“Now we arrive at the point where the **physicalist** approach I propose becomes different from the **standard formalist** philosophy of mathematics.”
Formal systems can be represented in physical systems.

We have access to a formal system only in some concrete physical representation.

Actually, there is nothing to be “represented”
I  Formal systems can be represented in physical systems

II We have access to a formal system only in some concrete physical representation

III Actually, there is nothing to be “represented”
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I. Formal systems can be represented in physical systems

II. We have access to a formal system only in some concrete physical representation

III. Actually, there is nothing to be "represented"
\[ p(e, x) = x \]
\[ (\forall x)(p(e, x) = x) \]
\[ (\forall x)(p(e, x) = x) \rightarrow p(e, e) = e \]
\[ p(e, e) = e \]
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\[ p(e, p(e, e)) = p(e, e) \]
\[ p(e) = e \to p(e, p(e, e)) = p(e, e) \to p(e, p(e, e)) = e \]
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\[ \{ \text{GroupTheory} \} \vdash p(e, p(e, e)) = e \]
group theory

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- variables: x, y, z, ...
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- function symbols: f, g
- predicate symbol: R
- others: (, ), ~

Derivation rules
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- \( \phi \rightarrow \forall \psi \phi \) (generalization)

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- \( \phi \rightarrow (\psi \rightarrow \phi) \)
- \( (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) \)
- \( (\sim \phi \rightarrow \psi) \rightarrow (\sim \psi \rightarrow \phi) \)
- \( (\forall \alpha \phi \rightarrow (\sim \psi \rightarrow \alpha)) \) (given that \( \alpha \) is not free in \( \psi \))
- \( (\forall \alpha \sim \psi \rightarrow \phi) \) (given that \( \alpha \) is not free in \( \psi \))
- \( (\forall \alpha \phi \rightarrow \alpha) \) (given that whenever a free occurrence of \( \alpha \) is replaced by \( \phi \), \( \alpha \) is free in \( \phi \))
- \( \alpha \equiv \alpha \)
- \( \forall \alpha \phi \rightarrow \phi' (\phi (w_1, w_2, ..., w_n) = \phi (w_1, w_2, ..., w_n)) \)
- \( \forall \alpha \phi \rightarrow \phi' (\phi (w_1, w_2, ..., w_n) = \phi (w_1, w_2, ..., w_n)) \)

Axioms II. (of group theory)
- \( \forall \phi \forall \psi \forall \chi \phi \rightarrow (\psi \rightarrow (\chi \rightarrow \psi)) \) (associative law)
- \( \forall \alpha \forall \chi \phi \rightarrow \phi (\alpha, \chi) = \chi \) (left identity)
- \( \forall \alpha \forall \chi \phi \rightarrow \phi (\alpha, \chi) = \alpha \) (left inverse)
Group theory

Physical system

\[ p(e, p(e, e)) = e \]
\[ p(e, p(e, e)) = e \]
“But **how can anybody know that**

\[ \{ \text{GroupTheory} \} \vdash p(e, p(e, e)) = e \]

**is true** in the formal system “in the mathematical sense”?
“But **how can anybody know that**

\[
\{\text{GroupTheory}\} \vdash p(e, p(e, e)) = e
\]

**is true** in the formal system “in the mathematical sense”? The usual idea is that **it can be known to anyone who executes the formal derivation of \( \varphi \) from GT in the head.** So what we actually do is **represent the formal system in a brain and observe the behavior of the brain.** The human brain is however not entirely reliable, so we prefer to execute the derivation in a brain+hand+pen +paper system.”
group theory

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\( p(e, p(e, e)) = e \)
\[
p(e, p(e, e)) = e
\]
\[ p(e, p(e, e)) = e \]

\[ p(e, p(e, e)) = e \]
“Upon what grounds can one physical representation be singled out as “the right one”? It is nonsense. We have to recognise that the only sources of our mathematical knowledge are the formal systems embodied in concrete physical forms; and this knowledge can be obtained only by a posteriori means.”
\[ p(e, p(e, e)) = e \]

\[ p(e, p(e, e)) = e \]

\[ p(e, p(e, e)) = e \]

\[ \{\text{GroupTheory}\} \vdash p(e, p(e, e)) = e \]
“Numbers, sets, groups and algebras have an autonomous reality quite independent of what the laws of physics decree, and the properties of these mathematical structures can be just as objective as Plato believed they were. But they are revealed to us only through the physical world. It is only physical objects, such as computers or human brains, that ever give us glimpses of the abstract world of mathematics.
“Numbers, sets, groups and algebras have an autonomous reality quite independent of what the laws of physics decree, and the properties of these mathematical structures can be just as objective as Plato believed they were. But they are revealed to us only through the physical world. It is only physical objects, such as computers or human brains, that ever give us glimpses of the abstract world of mathematics.

[...]

It seems that we have no choice but to recognize the **dependence of our mathematical knowledge** (though not, we stress, of mathematical truth itself) **on physics**, and that being so, it is time to abandon the classical view of computation as a purely logical notion independent of that of computation as a physical process.” - **David Deutsch**
I Formal systems can be represented in physical systems

II We have access to a formal system only in some concrete physical representation

III Actually, there is nothing to be “represented”
Platonism
platonic world
physical theory  

platonic world

L mathematics

matematika

physical theory  

platonic world

L mathematics

matematika
physical theory  ↔  platonic world

L  mathematics  S  semantics

platonic world

mathematics

semantics
physical world
physical world
physical world
The existence of a mathematical derivation, making a proposition of type

\[ L \vdash A \]

true, is a physical fact of the formal system as a part of the physical world. **To prove a theorem is** nothing but to observe this fact—for example, to observe a derivation process in a computer—that is, **to observe a physical fact about a physical system.**
Physicalism
"In order to explain the universal conviction that mathematical truths are necessary and certain, notice that there are many elements of our everyday knowledge which seem to be necessary and certain, but are actually obtained from inductive generalization. Break a long stick. We are “sure” about the outcome: the result is a shorter stick. This regularity of the physical world is known to us from experiences. The certainty of this knowledge is, however, no less than the certainty of the inference, say, from the Euclidean axioms to the height theorem."

"the certainty of mathematics, that is the degree of certainty with which one can know the result of a deductive inference, is the same as the degree of certainty of our knowledge about any other physical facts."
Mathematical monism
Max Tegmark's mathematical universe hypothesis goes further than full-blooded Platonism in asserting that not only do all mathematical objects exist, but nothing else does. Tegmark's sole postulate is: All structures that exist mathematically also exist physically. That is, in the sense that "in those [worlds] complex enough to contain self-aware substructures [they] will subjectively perceive themselves as existing in a physically 'real' world"
platonist
formalist
physicalist
mathematical monist