Quark model of hadrons and the SU(3) symmetry

David Nagy - particle physics (5)

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"Young man, if I could remember the names of these particles, I would have been a botanist."

Enrico Fermi to his student

Baryonic wave functions

$$\psi = \alpha(space) \cdot \beta(spin) \cdot \gamma(flavour) \cdot (\varepsilon(colour))$$

Since baryons are fermions, ψ must be antisymmetric for exchange of particles. Starting from

$$|\Delta^{++}\rangle = |uuu\rangle$$

(with isospin $I_3|uuu\rangle = \frac{3}{2}|uuu\rangle$ and charge $Q|uuu\rangle = 2|uuu\rangle$), we can get to Δ^+ by stepping down

$$I_-|uuu\rangle = \frac{1}{\sqrt{3}}(|uud\rangle + |udu\rangle + |duu\rangle) = |\Delta^+\rangle$$

$$I_-|\Delta^+\rangle = \frac{1}{\sqrt{3}}(|udd\rangle + |dud\rangle + |ddu\rangle) = |\Delta^0\rangle$$

$$I_-|\Delta^0\rangle = |ddd\rangle = |\Delta^-\rangle$$

If we step down diagonally we get Σ^{*+}

$$Q_-|uuu\rangle = \frac{1}{\sqrt{3}}(|uus\rangle + |usu\rangle + |suu\rangle) = |\Sigma^{*+}\rangle$$

$$I_-|\Sigma^{*+}\rangle = |\Sigma^{*0}\rangle$$

and so on. Altogether 10 states which are completely symmetric in flavour for exchange of particles ($\gamma(flavour)$ doesn't change sign). This is the baryon decuplet.

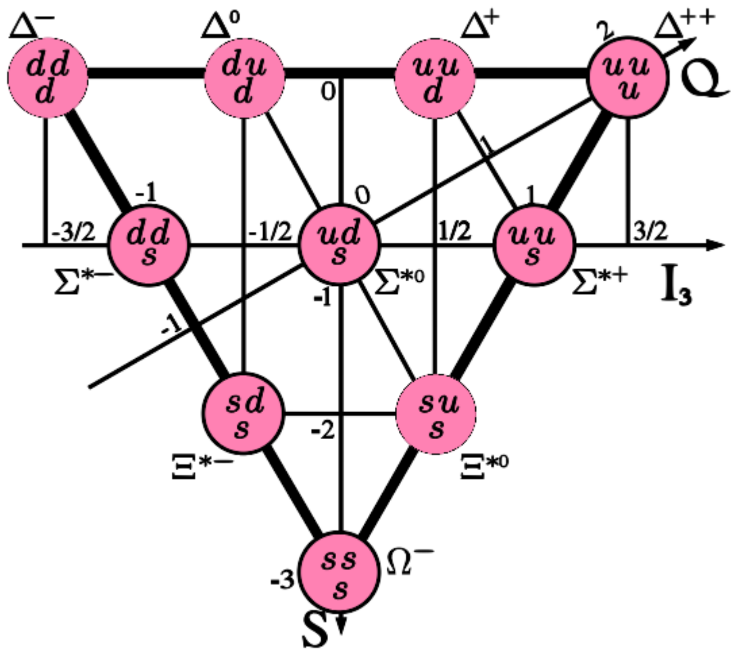

Figure 1: Baryon decuplet ($J = \frac{3}{2}$)

Altogether 8 states which are completely symmetric in $\beta(spin) \cdot \gamma(flavour)$ for exchange of particles (antisymmetric in both of them).

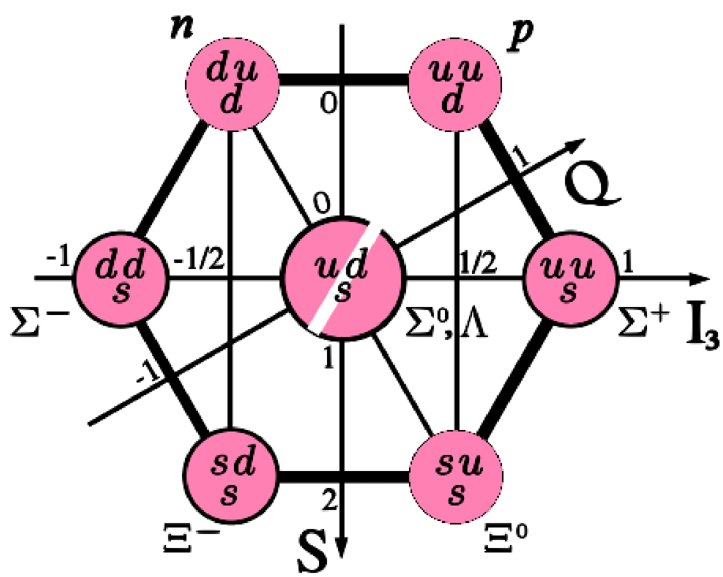

Figure 2: Baryon octet ($J = \frac{1}{2}$)

Pseudoscalar mesons

$$C \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$$

SU(3) symmetries

$$|q_i\rangle = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$$|q_i'\rangle = U_{\text{ij}}|q_i\rangle$$

where $\hat{U}$ is unitary with $det(\hat{U}) = 1$. The Gell-Mann matrices are a representation of the infinitesimal generators of the SU(3) group, thus any element of SU(3) can be written in the form

$$\hat{U} = e^{\text{i}\theta_a g_a}$$

where $\theta_a \in \mathbb{R}$, $a = 1, \ldots, 8$, $g_a = \frac{\lambda_a}{2}$ and the λ_a are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.$$

The matrices are traceless, hermitian and can be constructed from the commutation relations

$$[g_i, g_j] = i f^{ijk} g_k$$

where f^{ijk} are the structure constants with sum over k implied. Then a group element is i.e.

$$e^{\text{i}\theta\lambda_2 =} \begin{pmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here, $\{g_1, g_2, g_3\}$ constitutes a closed SU(2) subgalgebra. Thus, Isospin operator corresponds to $\hat{I}_i = g_i$ for $i = 1, 2, 3$

$$g_3|q\rangle = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} \frac{u}{2} \\ -\frac{d}{2} \\ 0 \end{pmatrix}$$

Hypercharge operator corresponds to $\hat{Y} = \frac{2}{\sqrt{3}} g_8$

$[\hat{Y}, \hat{I}_3] = 0$ within SU(3), and we can use their common eigenvalues to characterize states, with the eigenfunctions corresponding to particles.

(Light Quark) Mesons

Mesons are combinations of a quark and an antiquark $q\bar{q}$. Since the $\mathscr{H}$ space of a composite system is the tensor product of the state spaces of the component systems, and using the fusion rules we can decompose the tensor product of two representations of a group into a direct sum of irreducible representations

$$D_i \otimes D_j = \bigoplus_k N_{ij}^k D_k$$

we can decompose the composite system of two quarks (fundamental and conjugate representations corrsepond to quark and antiquark state space) as the direct sum of the trivial representation (singlet) and the adjoint representation (octet)

$$3 \otimes \bar{3} = 8 \oplus 1$$

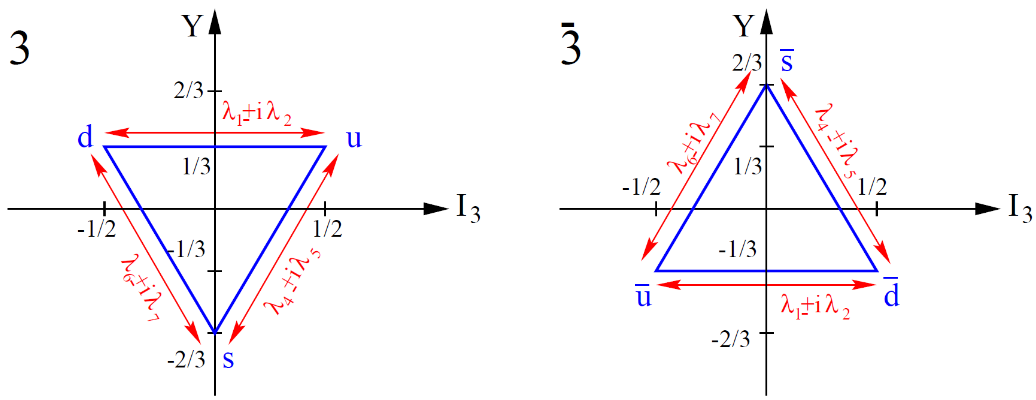

Figure 3: 3 and $\bar{3}$ triplets

(Light Quark) Baryons

Baryons are combinations of either three quarks qqq or three anti quarks $\overline{qqq}$ (antibaryons). Their decomposition is

$$3 \otimes 3 \otimes 3 = \underset{\text{resonances}}{1 \oplus 8} \oplus \underset{\text{octet}}{8} \oplus \underset{\text{decuplet}}{10}$$

In order to

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & \frac{-\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & \frac{-2\Lambda}{\sqrt{6}} \end{pmatrix}$$

Gell-Mann-Okubo mass formula (6)

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In case of exact SU(3) symmetry, all quark masses would have to be equal. In order to approximate the error from this, lets suppose that

1. $m_u = m_d < m_s$, and that
2. there is no other breaking of the symmetry.

Then

$$H_{strong} = H_0 + \tilde{H}$$

How does H˜ transform under SU(3)? For stationary quarks the rest energy and mass are equal

$$\langle q_i|H_{strong}|q_j\rangle = \langle q_i|M_{quark}|q_j.\rangle$$

Since

$$M_{quark} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

where $m_u = m_d$, so

$$\begin{aligned} \langle q_i|M_{quark}|q_j\rangle &= \begin{pmatrix} \frac{1}{3}(m_s+2m_u) & 0 & 0 \\ 0 & \frac{1}{3}(m_s+2m_u) & 0 \\ 0 & 0 & \frac{1}{3}(m_s+2m_u) \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{3}(m_u-m_s) & 0 & 0 \\ 0 & \frac{1}{3}(m_u-m_s) & 0 \\ 0 & 0 & \frac{2}{3}(m_u-m_s) \end{pmatrix} \\ &= \langle q_i|\left(\underbrace{\frac{1}{3}(m_s+2m_u)\,\hat{1}_{3\times 3}}_{SU(3)\ invariant} + \underbrace{\frac{(m_u-m_s)}{\sqrt{3}}\lambda_8}_{transforms\ as\ \lambda_8}\right)|q_j\rangle \end{aligned}$$

Let's suppose $\widetilde{H}$ is small. Then we can try to use first order perturbation

$$H_0|\psi_0\rangle = m_0|\psi_0\rangle$$

$$m_H = \langle\psi_0|H_{strong}|\psi_0\rangle = m_0 + \langle\psi_0|\widetilde{H}|\psi_0\rangle$$

In ψ_0 representation there is an F among F generators that transforms asλ_8

$$d_{8ab}F_aF_b = -\frac{1}{2\sqrt{3}}F_aF_a + \frac{\sqrt{3}}{2}(F_1^2+F_2^2+F_3^2) - \frac{1}{2\sqrt{3}}F_8^2$$

where

$$F_aF_a \propto \hat{1}$$

because of Schur's lemma

$$(F_1^2 + F_2^2 + F_3^2) = I(I+1)$$

$$\frac{1}{2\sqrt{3}} F_8^2 \propto Y$$

thus

$$\tilde{H} = \tilde{m_0}\hat{1} + \delta_{m_1} Y + \delta_{m_2} \left(I(I+1) - \frac{Y^2}{4} \right)$$

$$m_H = \tilde{m_0} + \delta_{m_1} Y + \delta_{m_2} \left(I(I+1) - \frac{Y^2}{4} \right)$$

Applications

For the baryon octet, there are 3 parameters and 4 masses, which means that they are not independent

$$2(m_N + m_\Xi) = 3m_\Lambda + m_\Sigma$$

For the decuplet, hadron mass is linear in strangeness

$$m_H = \tilde{m_0}' + \delta'_{m_1} Y$$

where we take $\tilde{m_0}' = \tilde{m_0} + 2\delta_{m_2}$ and $\delta'_{m_1} = \delta_{m_1} + \frac{3}{2}\delta_{m_2}$. The mass of Ω^- was predicted this way.
For the pseudoscalar octet

Paradoxes of the quark model, colour symmetry

1. there are no free quarks
2. there are no qq or $qqqq$ hadrons
3. the wave function of Δ^{++} is symmetric, because it's $\alpha(space)$ part is symmetric (otherwise there would be a particle with the same quantum numbers but smaller mass)

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